Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE SUM OF A CERTAIN TRIPLE SERIES.

BY ERNEST W. BROWN.

In obtaining the value of his infinite determinant,* Dr. G. W. Hill finds it necessary to obtain the sum of a series which is equivalent to

$f(\alpha) = \Sigma_i\Sigma_k\Sigma_{k'}\{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i+k+k'\}\{i+k+k'+1\}$,
where

$$\{j\} = \frac{1}{\alpha^2 - j^2}, \quad i = 0, \pm 1, \pm 2, \cdots; k, k' = 2, 3, \cdots,$$

α being a definite number not a positive or negative integer or zero. In his paper Hill develops the cases in which there are four factors with a simple or double summation and, with reference to $f(\alpha)$, remarks that it may be treated in an analogous manner and then gives its value. If we attempt to follow out Hill's method of partial fractions directly, the algebraic work becomes very heavy and, as some unsuccessful attempts by others appear to have been made in order to sum this particular series, it seems worth while to give in detail the method by which, after many trials, I have succeeded in showing that his expression for it is correct.

In the following developments Σ refers always to the complete sum of the expression to which it is attached unless one of the three letters i, k, k' is inserted, in which case it refers to the particular summation only.

It is to be remarked first that the series $\Sigma\{j\}$, $\Sigma\{j\}\{j'\}$, $\Sigma\{j\}\{j'\}\{j''\}$ are convergent and therefore remain so when each term is multiplied by a finite factor. Hence $f(\alpha)$ is convergent.

In all cases, since i has all integral values between $+\infty$ and $-\infty$, we can replace i by $i \pm$ any integer without altering the sum. Also, since k, k' have the same range, they may be interchanged. Finally,

$$\Sigma_k\{i+k+1\} = \Sigma_k\{i+k\} - \{i+2\}, \text{ etc.}$$

Put $-i-k-1$ for i. Then

$$f(\alpha) = \Sigma\{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i-k'\}\{i-k'+1\}.$$

We have, by partial fractions,

* On a part of the motion of the lunar perigee, etc., Cambridge, 1877 and Acta Math., vol. 8 (1886), pp. 1–36.

$$\{i+k\}\{i+k+1\} = \frac{1}{2\alpha}\cdot\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-k-1}+\frac{1}{\alpha+i+k}\right)$$
$$-\frac{1}{2\alpha}\cdot\frac{1}{2\alpha+1}\left(\frac{1}{\alpha-i-k}+\frac{1}{\alpha+i+k+1}\right)$$
$$=\frac{1}{2\alpha}\cdot\left(\frac{1}{2\alpha-1}-\frac{1}{2\alpha+1}\right)\left(\frac{1}{\alpha-i-k}+\frac{1}{\alpha+i+k}\right)$$
$$+\frac{1}{2\alpha}\cdot\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-k-1}-\frac{1}{\alpha-i-k}\right)$$
$$-\frac{1}{2\alpha}\cdot\frac{1}{2\alpha+1}\left(\frac{1}{\alpha+i+k+1}-\frac{1}{\alpha+i+k}\right).$$

Every term is finite and therefore $f(\alpha)$, with these two factors so expressed, is convergent. The first line of the latter expression is $2\{i+k\}/(4\alpha^2-1)$. As no other part of $f(\alpha)$ contains k, we can sum the two terms in the last line from $k = 2$ to $k = \infty$; the parts of these within the brackets give $-1/(\alpha-i-2)$ and $-1/(\alpha+i+2)$, respectively. Hence

$$\text{(1)}\quad \begin{aligned}\Sigma_k\{i+k\}\{i+k+1\} &= \frac{1}{4\alpha^2-1}[2\Sigma_k\{i+k\}-(2i+5)\{i+2\}]\\ &= \frac{1}{4\alpha^2-1}[2\Sigma_k\{i+k+1\}-(2i+3)\{i+2\}].\end{aligned}$$

Similarly

$$\text{(2)}\quad \Sigma_{k'}\{i-k'\}\{i-k'+1\} = \frac{1}{4\alpha^2-1}[2\Sigma_{k'}\{i-k'\}+(2i-1)\{i-1\}].$$

We have further,

$$-(2i+3)\{i+2\}\{i+1\} = \{i+1\}-\{i+2\},$$
$$(2i-1)\{i\}\{i-1\} = \{i\}-\{i-1\}.$$

Using these results in the product of (1) multiplied by $\{i+1\}$ and (2) multiplied by $\{i\}$, we obtain

$$f(\alpha) = \frac{1}{(4\alpha^2-1)^2}\Sigma[2\{i+1\}\{i+k+1\}+\{i+1\}-\{i+2\}]$$
$$\times[2\{i\}\{i-k'\}+\{i\}-\{i-1\}]$$
$$=\frac{1}{(4\alpha^2-1)^2}(4X+4B-4C+d_1-2d_2+d_3),$$

where

$$
\begin{aligned}
X &= \Sigma\{i\}\{i+1\}\{i+k+1\}\{i-k'\},\\
B &= \tfrac{1}{2}\Sigma\{i\}\{i+1\}[\{i+k+1\}+\{i-k'\}],\\
C &= \tfrac{1}{2}\Sigma\{i+1\}\{i-1\}\{i+k+1\}+\tfrac{1}{2}\Sigma\{i+2\}\{i\}\{i-k'\}\\
&= \tfrac{1}{2}\Sigma\{i\}\{i+2\}[\{i+k+2\}+\{i-k'\}],\\
&= \Sigma\{i\}\{i+2\}\{i-k'\} = \Sigma\{i\}\{i-2\}\{i+k\},
\end{aligned}
$$

by putting $-i-2$ for i and k' for k, in the former term, and

$$
d_1 = \Sigma\{i\}\{i+1\}, \quad d_3 = \Sigma\{i-1\}\{i+2\} = \Sigma\{i\}\{i+3),
$$

$$
d_2 = \tfrac{1}{2}\Sigma\{i\}\{i+2\}+\tfrac{1}{2}\Sigma\{i-1\}\{i+1\} = \Sigma\{i\}\{i+2\}.
$$

We have now to reduce X to a form in which there are three factors. Proceeding as before by partial fractions and separating into three parts, we have

$$
\begin{aligned}
X = {} & \frac{1}{2\alpha}\Sigma\left(\frac{1}{2\alpha-1}-\frac{1}{2\alpha+1}\right)\left(\frac{1}{\alpha-i}+\frac{1}{\alpha+i}\right)\{i+k+1\}\{i-k'\}\\
& +\frac{1}{2\alpha}\Sigma\left[\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-1}-\frac{1}{\alpha-i}\right)-\frac{1}{2\alpha+1}\left(\frac{1}{\alpha+i+1}-\frac{1}{\alpha+i}\right)\right]\\
& \qquad \{i+k+1\}\{i-k'\}.
\end{aligned} \tag{3}
$$

The first line of this gives $(2Y-2C)/(4\alpha^2-1)$ where

$$
Y = \Sigma\{i\}\{i+k+1\}\{i-k'\}+C = \Sigma\{i\}\{i+k\}\{i-k'\}.
$$

For the second line, we note that the double series

$$
\Sigma_i\Sigma_{k'}\frac{\{i+k+1\}\{i-k'\}}{\alpha-i-1} = \Sigma_i\Sigma_{k'}\left[\frac{\{i+k\}\{i-k'\}}{\alpha-i}-\frac{\{i+k\}\{i-2\}}{\alpha-i}\right]
$$

are convergent; the second form is obtained by putting $i-1$ for i and then $\Sigma_{k'}\{i-k'-1\} = \Sigma_{k'}\{i-k'\}-\{i-2\}$. Hence

$$
\begin{aligned}
& \Sigma_i\Sigma_{k'}\left(\frac{1}{a-i-1}-\frac{1}{\alpha-i}\right)\{i+k+1\}\{i-k'\}\\
& = \Sigma_i\Sigma_{k'}\left[\frac{\{i+k\}-\{i+k+1\}}{\alpha-i}\{i-k'\}\right]-\Sigma_i\frac{\{i+k\}\{i-2\}}{\alpha-i}.
\end{aligned} \tag{4}
$$

If now we sum for values of k, the left hand member is convergent and the first term of the right hand member gives a series whose typical term is $\phi(k)-\phi(k+1)$, where $\phi(k)$ is a rational fractional function of k of one

degree less in the numerator than in the denominator. The series is therefore convergent and has a sum $\phi(2)$. Hence (4) is equal to

$$\Sigma_i \Sigma_{k'} \frac{\{i+2\}\{i-k'\}}{\alpha - i} - \Sigma_i \Sigma_k \frac{\{i-2\}\{i+k\}}{\alpha - i} = -2\Sigma i\{i\}\{i-2\}\{i+k\},$$

by putting $-i$ for i, k' for k in the first term.

This gives the first half of the second line of (3); the second half is obtained from it by changing the sign of α. Adding the two portions together we obtain

$$X = \frac{2}{4\alpha^2 - 1}(Y - C) - \Sigma \frac{4i}{4\alpha^2 - 1}\{i\}\{i-2\}\{i+k\}.$$

But

$$-4i = (\alpha^2 - i^2) - [\alpha^2 - (i-2)^2] - 4.$$

Hence,

$$X = \frac{1}{4\alpha^2 - 1}(2Y - 2C + \Sigma\{i-2\}\{i+k\} - \Sigma\{i\}\{i+k\} - 4C)$$

$$= \frac{1}{4\alpha^2 - 1}(2Y - 6C + \Sigma\{i\}\{i+k+2\} - \Sigma\{i\}\{i+k\})$$

$$= \frac{1}{4\alpha^2 - 1}(2Y - 6C - d_2 - d_3).$$

Again

$$Y = \Sigma\{i\}[\{i+k-1\} - \{i+1\}][\{i-k'+1\} - \{i-1\}]$$

$$= Z - 2A + d_4,$$

where

$$Z = \Sigma\{i\}\{i+k-1\}\{i-k'+1\},$$

$$A = \tfrac{1}{2}\Sigma[\{i\}\{i+1\}\{i-k'+1\} + \{i\}\{i-1\}\{i+k-1\}]$$

$$= \tfrac{1}{2}\Sigma\{i\}\{i-1\}[\{i-k\} + \{i+k-1\}],$$

by putting $i - 1$ for i and k for k' in the first term, and

$$d_4 = \Sigma\{i\}\{i-1\}\{i+1\}.$$

Reduction of A, B, C.—Suppose we desire to sum the expression denoted by A with respect to k. We note that the portion in square brackets contains all the functions $\{i + j\}$ where j goes from $-\infty$ to $+\infty$, except those for $j = -1, 0$, that is, all the functions $\{j\}$ except $\{i-1\}$, $\{i\}$. Hence

$$2A = \Sigma_i \Sigma_j \{i\}\{i-1\}\{j\} - \Sigma\{i\}\{i-1\}^2 - \Sigma\{i\}^2\{i-1\}.$$

But $\Sigma_j\{j\} = \pi \cot \pi\alpha/\alpha$. Therefore, putting $i + 1$ for i in the last term and changing the sign of i throughout, we obtain

$$A = \frac{\pi \cot \pi\alpha}{2\alpha} d_1 - d_5,$$

where

$$d_5 = \Sigma\{i\}\{i+1\}^2.$$

Similarly, putting k for k',

$$2B = \Sigma\{i\}\{i+1\}\{j\} - 2\{i\}\{i+1\}\{i-1\} - 2\{i\}\{i+1\}^2,$$

$$B = \frac{\pi \cot \pi\alpha}{2\alpha} d_1 - d_4 - d_5.$$

Also, with the second form for C, after putting k' for k,

$$C = \frac{\pi \cot \pi\alpha}{2\alpha} d_2 - d_6 - d_7 - \tfrac{1}{2} d_4,$$

where

$$d_6 = \Sigma\{i\}\{i+2\}\{i-1\}, \quad d_7 = \Sigma\{i\}\{i+2\}^2.$$

Gathering together the results so far obtained, we find

$$f(\alpha) = \frac{4}{(4\alpha^2-1)^3}(2Z - 4A - 6C - d_2 - d_3 + 2d_4) + \frac{1}{(4\alpha^2-1)^2}(4B - 4C + d_1 - 2d_2 + d_3)$$

$$(5) \qquad = \frac{4}{(4\alpha^2-1)^3}\left[2Z - \frac{\pi \cot \pi\alpha}{\alpha}(2d_1 + 3d_2) - d_2 - d_3 + 5d_4 + 4d_5 + 6d_6 + 6d_7\right] + \frac{1}{(4\alpha^2-1)^2}\left[\frac{2\pi \cot \pi\alpha}{\alpha}(d_1 - d_2) + d_1 - 2d_2 + d_3 - 2d_4 - 4d_5 + 4d_6 + 4d_7\right].$$

All that remains now is to find the sums of the various series d_1, d_2, $\cdots$ with respect to i, and that of Z with respect to i, k, k'.

Value of Z.—Let p, q, r, $\cdots$ be any quantities and let S_1p be their sum, S_2pq the sum of their products taken two at a time, S_3pqr the sum of their products taken three at a time. We have

$$(S_1p)^3 = S_1p^3 + 3S_2p^2q + 6S_3pqr$$
$$= S_1p^3 + 3(S_1p^2)(S_1p) - 3S_1p^3 + 6S_3pqr,$$

therefore

$$S_3pqr = \tfrac{1}{6}S_1p^3 - \tfrac{1}{2}(S_1p^2)(S_1p) + \tfrac{1}{3}S_1p^3.$$

In order to apply this to the computation of Z we note that $i + k - 1$ represents, for integral values of k from 2 to ∞, all integers greater than i, and $i - k' + 1$, for a similar range of k', all integers less than i. Since i has the range $+\infty$ to $-\infty$, Z is the sum of the products, taken three at a time, of the functions $\{i\}$. We therefore have

$$Z = \tfrac{1}{6}(\Sigma\{i\})^3 - \tfrac{1}{2}(\Sigma\{i\}^2)(\Sigma\{i\}) + \tfrac{1}{3}\Sigma\{i\}^3.$$

Now

$$\Sigma\{i\} = \Sigma\frac{1}{\alpha^2 - i^2} = \frac{\pi\cot\pi\alpha}{\alpha}.$$

Differentiating this twice with respect to α, we obtain

$$\Sigma\{i\}^2 = \frac{\pi^2}{2\alpha^2}\cdot\frac{1}{\sin^2\pi\alpha} + \frac{\pi\cot\pi\alpha}{2\alpha^3},$$

$$\Sigma\{i\}^3 = \frac{\pi^3\cot\pi\alpha}{4\alpha^3\sin^2\pi\alpha} + \frac{3}{8}\frac{\pi^2}{\alpha^4\sin^2\pi\alpha} + \frac{3}{8}\frac{\pi\cot\pi\alpha}{\alpha^5},$$

whence

$$Z = \frac{\pi\cot\pi\alpha}{8\alpha^5} - \frac{\pi^2(2\cos^2\pi\alpha - 1)}{8\alpha^4\sin^2\pi\alpha} - \frac{1}{6}\frac{\pi^3\cot\pi\alpha}{\alpha^3}$$

$$= \frac{\pi\cot\pi\alpha}{\alpha}\left[\frac{1}{8\alpha^4} - \frac{\pi\cot 2\pi\alpha}{4\alpha^3} - \frac{\pi^2}{6\alpha^2}\right].$$

Values of d_1, d_2, d_3—We need $\{i\}\{i+k\}$ where k is a given integer. We have

$$\Sigma_i\{i\}\{i+k\} = \Sigma_i\frac{4}{(2\alpha+k)^2 - (2i+k)^2}\cdot\frac{4}{(2\alpha-k)^2 - (2i+k)^2}$$

$$= \frac{2}{\alpha k}\Sigma_i\left[\frac{1}{(2\alpha-k)^2 - (2i+k)^2} - \frac{1}{(2\alpha+k)^2 - (2i+k)^2}\right]$$

$$= \frac{\pi\cot\pi\alpha}{\alpha k(2\alpha-k)} - \frac{\pi\cot\pi\alpha}{\alpha k(2\alpha+k)} = \frac{2\pi\cot\pi\alpha}{\alpha(4\alpha^2 - k^2)}.$$

Hence

$$(4\alpha^2 - 1)\Sigma_i\{i\}\{i+k\} = \left(1 + \frac{k^2 - 1}{4\alpha^2 - k^2}\right)\frac{2\pi\cot\pi\alpha}{\alpha}.$$

Values of d_4, d_6.—We have, if $k \neq k'$, by a similar proceedure,

$$\begin{aligned}\{i\}\{i+k\}\{i+k'\} = \frac{1}{2\alpha(k-k')}\Bigg[&\frac{1}{\alpha^2 - i^2}\cdot\frac{4}{(2\alpha - k + k')^2 - (2i + k + k')^2}\\ &- \frac{1}{\alpha^2 - i^2}\cdot\frac{4}{(2\alpha + k - k')^2 - (2i + k + k')^2}\Bigg].\end{aligned} \tag{6}$$

By partial fractions, the second term within the square brackets is equal to

$$\frac{A}{\alpha+i} + \frac{B}{\alpha-i} + \frac{C}{\alpha+i+k} + \frac{D}{\alpha-i-k'},$$

where

$$A = -\frac{1}{2\alpha k}\cdot\frac{1}{2\alpha - k'}, \qquad B = \frac{1}{2\alpha k'}\cdot\frac{1}{2\alpha + k},$$

$$C = \frac{1}{k(2\alpha + k)} \cdot \frac{1}{2\alpha + k - k'}, \quad D = -\frac{1}{k'(2\alpha - k')} \cdot \frac{1}{2\alpha + k - k'}.$$

If we add the corresponding term obtained by changing the sign of i we find convergent series. We can therefore write

$$\Sigma_i \frac{1}{\alpha + i + \lambda} = \pi \cot \pi\alpha,$$

where λ is an integer, and the second term of (6) is equal to $(A + B + C + D)\pi \cot \pi\alpha$. The first term of (6) within the square brackets is the same quantity with the sign of α and the sign changed. Adding, we find at once,

$$\Sigma_i \{i\}\{i + k\}\{i + k'\} = \left[\frac{1}{k'(k - k')} \cdot \frac{1}{4\alpha^2 - k^2} - \frac{1}{k(k - k')} \cdot \frac{1}{4\alpha^2 - k'^2} - \frac{1}{kk'} \cdot \frac{1}{4\alpha^2 - (k - k')^2}\right] \frac{2\pi \cot \pi\alpha}{\alpha}.$$

Values of d_5, d_7.—We have, by partial fractions,

$$\{i\}\{i + k\}^2 = \frac{A}{\alpha + i} + \frac{B}{\alpha - i} + \frac{C_1}{\alpha + i + k} + \frac{D_1}{\alpha - i - k} + \frac{C_2}{(\alpha + i + k)^2} + \frac{D_2}{(\alpha - i - k)^2},$$

where

$$A = \frac{1}{2\alpha k^2} \cdot \frac{1}{(2\alpha - k)^2}, \quad C_2 = -\frac{1}{4\alpha^2 k} \cdot \frac{1}{2\alpha + k},$$

$$B = \frac{1}{2\alpha k^2} \cdot \frac{1}{(2\alpha + k)^2}, \quad D_2 = \frac{1}{4\alpha^2 k} \cdot \frac{1}{2\alpha - k}.$$

If $i = -k$, we have

$$\frac{A}{\alpha - k} + \frac{B}{\alpha + k} + \frac{C_1 + D_1}{\alpha} + \frac{C_2 + D_2}{\alpha^2} = \frac{1}{\alpha^4} \cdot \frac{1}{\alpha^2 - k^2}.$$

Thence

$$\alpha(A + B + C_1 + D_1) = \frac{\alpha k}{\alpha + k} B - \frac{\alpha k}{\alpha - k} A - (C_2 + D_2) + \frac{1}{\alpha^2(\alpha^2 - k^2)}$$

$$= -\frac{8\alpha^2 + k^2}{(4\alpha^2 - k^2)^2(\alpha^2 - k^2)} - \frac{1}{2\alpha^2(4\alpha^2 - k^2)} + \frac{1}{\alpha^2(\alpha^2 - k^2)}$$

$$= \frac{4}{(4\alpha^2 - k^2)^2} + \frac{2}{k^2(4\alpha^2 - k^2)} - \frac{1}{2\alpha^2 k^2}.$$

But, as before,

$$\Sigma \frac{1}{\alpha + i + \lambda} = \pi \cot . \pi\alpha, \quad \Sigma \frac{1}{(\alpha + i + \lambda)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}.$$

Hence

$$\Sigma_i\{i\}\{i+k\}^2 = \left[\frac{4}{(4\alpha^2-k^2)^2}+\frac{2}{k^2(4\alpha^2-k^2)}-\frac{1}{2\alpha^2k^2}\right]\frac{\pi\cot\pi\alpha}{\alpha} + \frac{\pi^2}{(4\alpha^2-k^2)2\alpha^2\sin^2\pi\alpha}$$

and

$$(4\alpha^2-1)\Sigma_i\{i\}\{i+k\}^2 = \left[\frac{4(k^2-1)}{(4\alpha^2-k^2)^2}+\frac{6k^2-2}{k^2(4\alpha^2-k^2)}+\frac{1}{2\alpha^2k^2}\right]\frac{\pi\cot\pi\alpha}{\alpha} + \left[1+\frac{k^2-1}{(4\alpha^2-k^2)}\right]\frac{\pi^2}{2\alpha^2\sin^2\pi\alpha}.$$

Final value of $f(\alpha)$.—All the quantities in (5) can now be obtained by inserting the special values of k to find $d_1, \cdots, d_7$. Each term of the second line of (5) within the square brackets is to be multiplied by $4\alpha^2-1$ so that $(4\alpha^2-1)^{-3}$ appears as a factor of $f(\alpha)$. We also put

$$\frac{1}{\sin^2\pi\alpha}=\frac{2\cot\pi\alpha}{\sin 2\pi\alpha},\quad \cot^2\pi\alpha=\frac{1+\cos 2\pi\alpha}{\sin 2\pi\alpha}\cot\pi\alpha,$$

so that $\cot\pi\alpha$ also appears as a factor of $f(\alpha)$. We obtain, finally,

$$f(\alpha) = \frac{\pi\cot\pi\alpha}{\alpha(4\alpha^2-1)^3}\left[-\frac{4\pi^2}{3\alpha^2}+\frac{\pi\cot 2\pi\alpha}{\alpha}\left(-\frac{2}{\alpha^2}-\frac{16}{4\alpha^2-1}-\frac{36}{4\alpha^2-4}\right)\right.$$
$$\left.+\frac{1}{\alpha^4}-\frac{25}{2}\cdot\frac{1}{\alpha^2}+\frac{64}{(4\alpha^2-1)^2}-\frac{32}{4\alpha^2-1}+\frac{144}{(4\alpha^2-4)^2}+\frac{18}{4\alpha^2-4}+\frac{64}{4\alpha^2-9}\right].$$

Hill's expression is derived from $f(\alpha)$ by replacing $4\alpha^2$ by Θ_0 and dividing by 2^{12}: this is the coefficient of $-\Theta_1^6$ in the expansion of his infinite determinant. When these changes are made, the above result agrees with that of Hill.

Some easy tests of its accuracy are available from the fact that $\alpha = \pm\frac{1}{2}$, $\alpha = \pm\frac{3}{2}$ cannot make $f(\alpha)$ infinite. If then we put $\alpha = \pm\frac{1}{2}+x$, $\alpha = \pm\frac{3}{2}+x$, and expand in powers of x, all negative powers must disappear. Further, since α does not occur in any term of the original expression for $f(\alpha)$ to a power higher than its square, it follows that $\lim \alpha^2 f(\alpha)$, $\alpha \doteq 0$, must be finite. Similarly $\lim(\alpha\pm 1)f(\alpha)$, $\alpha \doteq \mp 1$, must be finite. The expression has been found to satisfy all these tests.*

*Note added Jan. 24, 1912. It should be stated that J. C. Adams had also computed this series and had given the result in a different form (Mon. Not. Roy. Astr. Soc., vol. 48, p. 47), but with no indication of his method of procedure. An inspection of the extracts from his manuscripts (Coll. Works, vol. 2, p. 92), made since this paper was completed, shows that he followed methods similar to those adopted here.

New Haven, Conn.,
October 10, 1911.